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ON A MEASURE OF THE CLOSENESS OF NEUTRAL SYSTEMS TO INTERNAL RESONANCE*

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For certain classes of parametrically perturbed resonance systems that are neutral in a linear approximation, a quantitative characteristic is introduced for the closeness of the system of resonance: the magnitude of the critical detuning value for resonance δ^* at which the change in stability occurs as the system withdraws from resonance. The problem of finding this critical value is made complicated by the non-linear nature of the change in stability in neutral systems. It is solved below for third-order resonances in a situation that guarantees the passage of instability into asymptotic stability as the system withdraws from resonance.

Knowledge of the quantity δ^* enables the strong instability domain /1, 2/ in parameter space to be estimated, enables the danger of resonance to be characterized, and enables the structural parameter in the system, the shift of the resonance phases, to be clarified, whose variation would enable the danger of resonance to be increased or reduced.

1. Formulation of the problem. Fundamental assumptions. In the l-dimensional real space R^l we consider the system of differential equations that depends continuously on the parameter $\mu \in D$

$$z^{\cdot} = A(\mu) z + \sum_{j=k-1\geqslant 2}^{\infty} F^{(j)}(\mu, t, z) \tag{1.1}$$

where $D \subseteq R^d$ is a certain closed d-dimensional domain containing the origin, and $F^{(j)}$ are l-dimensional vector forms of j-th order whose coefficients are almost periodic functions of t uniformly in $\mu \in D$.

Let the matrix $A(\mu)$ have n pairs of different purely imaginary eigenvalues $\pm i\nu_s(\mu), s = 1, \ldots, n$ in D while the remaining eigenvalues have negative real parts in D.

Retaining the definitions from /3/, we consider (1.1) to be an F-system and there is a k-th -order resonance therein for $\mu = 0$ ($m_s \geqslant 0$ are integers):

$$\lambda = \langle m, \nu(0)\rangle \in N_2'^k, \quad m = (m_1, \ldots, m_n), \quad k = |m| = m_1 + \ldots + m_n. \tag{1.2}$$

The concepts of the F-system and the set $N_2'^k$ are described in detail in /3/. We recall that the continuous normal form of F-systems is reducible to autonomous form while $N_2'^k$ is contained in the minimum modulus generated by the spectrum of the non-linearity coefficients.

We will confine ourselves to studying a purely critical system when $l = 2n$. The case $l > 2n$ reduces to it by using the reduction principle /4/.

In addition to the initial parameters it is convenient to introduce the parameters $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ and the resonance detuning δ by setting

$$\varepsilon_s(\mu) = \nu_s(\mu) - \nu_s(0), \quad \delta(\mu) = \langle m, \varepsilon(\mu)\rangle$$

The equation $\delta = 0$ defines a certain k-resonance surface Γ_k in D.

The k-resonance surface is mapped into a k-resonance plane Π_k: $\langle m, \varepsilon(\mu)\rangle = 0$ in the

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space of the parameters ε. The quantity $|\delta|$ characterizes the distance to the plane Π_k. We shall henceforth consider that the vector μ has the dimensionality $d \geqslant n$ and the image of the mapping $\varepsilon(\Gamma_k)$ is a certain neighbourhood of zero in the subspace defined by the plane Π_k.

Condition R. There are no q-resonance planes with $q \leqslant k+1$ different from Π_k in ε-space.

Later assumptions substantially utilize the continuous and ordinary normal form of the F-systems /3/. Let us execute a continuous normalization of the F-system (1.1) in D. Writing explicitly the smallest terms of the internal and identity resonances, we obtain

$$u_s^{\cdot} = i\nu_s u_s + \alpha_s e^{i\lambda t} \bar{u}^{m-\delta_s} + u_s \sum_{|p|=N} \alpha_p^{(s)} \omega^p + O(\|u\|^{2N+2}) \tag{1.3}$$

$N = [k/2]$, $s = 1, 2, \ldots, n$, $\omega = (\omega_1, \ldots, \omega_n)$ $\omega_s = |u_s|^2$, $\omega^p = \omega_1^{p_1} \ldots \omega_n^{p_n}$ (δ_s is the direction in R^n). The substitution $u_s = v_s \exp(i\nu_s(0)t)$ transfers (1.3) into a system autonomous to the $(2N+2)$-th order.

$$v_s^{\cdot} = i\varepsilon_s v_s + \alpha_s \bar{v}^{m-\delta_s} + v_s \sum_{|p|=N} \alpha_p^{(s)} \omega^p + O(\|v\|^{2N+2}) \tag{1.4}$$

We now consider the domain D^* which is obtained by removal of the k-resonance surface Γ_k from D. The ordinary normal form of the F-system (1.1) in D^* will not contain internal resonance terms. It can be obtained by annihilating these terms in (1.4). The necessary transformation has the form

$$v_s = v_s^* + i\alpha_s \delta^{-1} v^{*m-\delta_s} \tag{1.5}$$

which takes (1.4) over into the system

$$v_s^{\cdot *} = i\varepsilon_s v_s^* + v_s^* \sum_{|p|=N} \alpha_p^{(s)*} \omega^{*p} + O(\|v^*\|^{2N+2}) \tag{1.6}$$

It is seen from (1.5) that certain coefficients of the non-linear terms in (1.6) that depend on δ^{-1} increase without limit as the system approaches the resonance surface $(\delta \to 0)$. It can be established that there already are such coefficients to the $(2k-3)$-th order. For $k=3$ they already are manifest to third order. Hence, the problem under consideration has a singularity for $k \geqslant 3$.

We will confine ourselves to studying the case $k=3$. Systems (1.4) and (1.6) take the form

$$v_s^{\cdot} = i\varepsilon_s v_s + \alpha_s \bar{v}^{m-\delta_s} + v_s \sum_{j=1}^{n} \alpha_{sj} \omega_j + O(\|v\|^4) \tag{1.7}$$

$$v_s^{\cdot *} = i\varepsilon_s v_s^* + v_s^* \sum_{j=1}^{n} \alpha_{sj}^* \omega_j^* + O(\|v^*\|^4) \tag{1.8}$$

The interrelation between the coefficients of both normal forms, which has been set up using (1.5), plays a substantial part in the analysis. Omitting calculations, we present values of n, m, λ, δ and the connection formulas for each kind of third-order resonance.

One-frequency resonance

$$n = 1, \quad m = 3, \quad \lambda = 3\nu_1(0), \quad \delta = 3\varepsilon_1, \quad \alpha_{11}^* = \alpha_{11} - i\delta^{-1}\alpha_1\bar{\alpha}_1 \tag{1.9}$$

Two-frequency resonance

$$n = 2, \; m = (1, 2), \; \lambda = \nu_1(0) + 2\nu_2(0), \; \delta = \varepsilon_1 + 2\varepsilon_2 \tag{1.10}$$

$$\alpha_{11}^* = \alpha_{11}, \quad \alpha_{12}^* = \alpha_{12} - 2i\delta^{-1}\alpha_1\bar{\alpha}_2$$

$$\alpha_{21}^* = \alpha_{21} - i\delta^{-1}\alpha_2\bar{\alpha}_2, \quad \alpha_{22}^* = \alpha_{22} - i\delta^{-1}\alpha_2\bar{\alpha}_1$$

Three-frequency resonance

$$n = 3, \quad m = (1, 1, 1), \quad \lambda = \nu_1(0) + \nu_2(0) + \nu_3(0)$$

$$\delta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3, \quad \alpha_{ss}^* = \alpha_{ss}$$

$$\alpha_{sj}^* = \alpha_{sj} - i\delta^{-1}\alpha_s\bar{\alpha}_r, \quad j, s, r = 1, 2, 3, \quad j \neq s \neq r \tag{1.11}$$

We consider the F-system (1.1). If $\delta = 0$, then the system is strictly resonant. As $|\delta|$ increases system (1.1) converges with the resonance surface. For a sufficiently broad class of systems, as $|\delta|$ grows it is possible for a time to appear such that for $\delta = \delta^*$ a change in stability first occurs. We call δ^* the critical value of resonance **detuning**. We will introduce additional assumptions for which the problem of determining δ^* will be solved.

Condition N. The F-system (1.1) is unstable in a second approximation on the resonance surface Γ_3 $(\delta = 0)$.

Condition U. The F-system (1.1) is asymptotically stable for $\delta = \infty$ (far from resonance and for $\forall \mu \in D$.

Both these conditions ensure the presence of the needed situation: the passage of the instability into asymptotic instability $(H \to AY)$ as $|\delta|$ increases (we do not consider the passage $AY \to H$ here). The results obtained in /5/ enable us to write the conditions N and U in the language of the normal form coefficients in each of the cases of one-, two-, and three-frequency resonance.

Remark. Condition U means that if the internal resonance terms are deleted in the continuous normal form, then what remains in the system will be asymptotically stable in D. Indeed, by setting $\delta^{-1} = 0$, it is seen form Eqs.(1.9)-(1.11) connecting both types of normalization that then $\alpha_{sj}^* = \alpha_{sj}$.

2. One-frequency resonance. From an examination of (1.7) on Γ_3 it follows that system (1.1) is unstable for $\alpha_1 \neq 0$. This fact is established exactly as in the periodic case /6/. Condition N reduces to the requirement $\alpha_1 \neq 0,\ \forall \mu \in \Gamma_3$.

Furthermore, considering (1.8), we establish that condition U reduces to the requirement $\operatorname{Re} \alpha_{11} < 0,\ \forall \mu \in D$. It is seen from (1.9) that $\operatorname{Re} \alpha_{11}^* = \operatorname{Re} \alpha_{11}$. This equality shows that when the condition $\alpha_1 \neq 0,\ \operatorname{Re} \alpha_{11} < 0$ are satisfied, the critical value of resonance detuning is $\delta^* = 0$. In other words, the influence of one-frequency resonance is weak: it does not alter the asymptotic stability property in any proximity to resonance by making the system unstable only for strict resonance.

3. Two-frequency resonance (the general case). We consider the system (1.7) and we introduce the notation $A = \operatorname{Im} \alpha_1 \bar{\alpha}_2,\ a_{sj} = \operatorname{Re} \alpha_{sj},\ a_{sj}^* = \operatorname{Re} \alpha_{sj}^*,\ \sigma = a_{11} + 2a_{21},\ \Delta = a_{11}a_{22} - a_{21}a_{12}$. We will assume the conditions of the general case to be satisfied /2/:

$$a_{11} \neq 0, \quad \sigma \neq 0, \quad A \neq 0 \quad (\forall \mu \in \Gamma_3)$$

Making conditions N and U specific. We consider system (1.7) on Γ_3. In the general case condition N is satisfied automatically: $A \neq 0$. Indeed, for $A \neq 0$ system (1.7) is unstable $\forall \mu \in \Gamma_3$ (see Theorem 2.1 in /2/).

We consider system (1.8) for the formulation of the condition U. In conformity with Theorem 2.2 in /2/, for its asymptotic stability it is necessary and sufficient that there exist such $\gamma_1, \gamma_2 > 0$ that the quadratic form

$$\gamma_1 a_{11}^* \omega_1^{*2} + (\gamma_1 a_{12}^* + \gamma_2 a_{21}^*) \omega_1^* \omega_2^* + \gamma_2 a_{22}^* \omega_2^{*2}$$

is negative-definite in the cone $\omega_{1,2}^* \geqslant 0$. For this, in turn, it is necessary and sufficient that one of the following two groups of conditions be satisfied:

α^*: 1) $a_{11} < 0$; 2) $a_{22} - A\delta^{-1} < 0$; 3) $a_{12} + 2A\delta^{-1} < 0$ or $a_{21} < 0$; β^*: 1) $a_{11} < 0$; 2) $a_{22} - A\delta^{-1} < 0$; 3) the third condition α^* is violated but $\Delta > \sigma A \delta^{-1}$.

If the ratio $A\delta^{-1}$ is neglected, we obtain the condition U which requires satisfaction of one of the following groups of conditions:

$$\alpha\text{: 1) } a_{11} < 0;\ \text{2) } a_{22} < 0;\ \text{3) } a_{12} < 0 \text{ or } a_{21} < 0;$$
$$\beta\text{: 1) } a_{11} < 0;\ \text{2) } a_{22} < 0;\ \text{3) } a_{12} > 0,\ a_{21} > 0,\ \Delta > 0$$

Formulation of the result. To find δ^* four logically possible cases are examined that are related to the satisfaction of one of the conditions α or β for a fixed sign of the quantity $A\delta$. The values found for $\delta^* \neq 0$ that include all cases of the transition $H \to AY$ as $|\delta|$ increases are represented in Table 1.

Table 1

α	1°	$A\delta < 0$	$\lvert\delta^*\rvert = \lvert A a_{22}^{-1}\rvert$
	2°	$A\delta > 0,\ a_{21} > 0,\ \sigma > 0$	$\lvert\delta^*\rvert = \min\{2\lvert A\rvert a_{21}^{-1},\ \sigma A \delta^{-1}\}$
β	3°	a) $A\delta < 0,\ \sigma > 0$ б) $A\delta < 0,\ \sigma < 0,\ 2\lvert a_{22}\rvert \geqslant a_{12}$	$\lvert\delta^*\rvert = \lvert A a_{22}^{-1}\rvert$
	4°	в) $A\delta < 0,\ \sigma < 0,\ 2\lvert a_{22}\rvert < a_{12}$ г) $A\delta > 0,\ \sigma > 0$	$\lvert\delta^*\rvert = \lvert\delta A \Delta^{-1}\rvert$

In all the remaining cases of satisfying conditions α or β that are not included in Table 1, $\delta^* = 0$. In these cases $A\delta > 0$ always. System (1.1) behaves as follows: the asymptotic stability for $A\delta > 0$ is replaced by instability for $\delta = 0$, which is conserved for $A\delta < 0$ until $|\delta| < |\delta^*|$, where δ^* (for $A\delta < 0$) can be determined from the appropriate line in Table 1.

Finding δ^*. If the conditions α (β) are satisfied, then for sufficiently large $|\delta|$ the condition α^* (β^*) are satisfied and system (1.1) is asymptotically stable far from resonance for finite values of δ. As $|\delta|$ diminishes, the instability occurs for those values of δ

when at least one of the inequalities in conditions α^* or β^* is violated. As is seen from (1.10), as δ changes only the second and third inequalities in conditions α^* and β^* can be spoiled. Finding δ^* requires the determination of the least values of $|\delta|$ for which at least one of these inequalities is first violated.

It is convenient to perform the analysis in the following four cases: 1°. α and $A\delta < 0$; 2°. α and $A\delta > 0$; 3°. β and $A\delta < 0$; 4°. β and $A\delta > 0$.

We present it just for the case 3° (the analysis is analogous in the remaining cases).

Thus, let the condition β be satisfied and $A\delta < 0$. As $|\delta|$ diminishes the instability can occur only because of violation of the second and third conditions of β^*. The second condition of β^* is violated for $|\delta| < |\delta_1| \equiv |Aa_{22}^{-1}|$. Let us analyse the third condition in β^*. Violation of the inequality $a_{12} + 2A\delta^{-1} > 0$ before the second inequality of β^* takes the conditions β^* over into the conditions α^* and for such $|\delta| > |\delta_1|$ the system remains asymptotically stable. *When the second condition of β^* is satisfied instability can set in* only for those δ for which the following inequalities are simultaneously true:

$$a_{12} + 2A\delta^{-1} > 0, \quad \Delta < \sigma A\delta^{-1} \tag{3.1}$$

Taking into account that $a_{12} > 0$, $A\delta < 0$, $\Delta > 0$, we see that system (3.1) is common just for

$$\sigma < 0, \quad a_{12} > 2|a_{22}| \tag{3.2}$$

and the system of inequalities (3.1) can be rewritten in the form

$$|\delta_2| \equiv 2|A|\,|a_{12}|^{-1} < |\delta| < |\sigma A|\,\Delta^{-1} \equiv \delta_3 \tag{3.3}$$

Comparing $|\delta_1|$ with $|\delta_2|$ and $|\delta_3|$ we see that the inequality $|\delta_2| < |\delta_1| < |\delta_3|$ in conditions (3.2) is true. It hence follows that we have $|\delta^*| = |\delta_3|$ for the satisfaction of conditions (3.2) and $|\delta^*| = |\delta_1|$ for the violation of these conditions (cases a), b), c) in Table 1).

4. Three-frequency resonance (general cases). We consider system (1.7) for $n = 3$. We again use the notation $a_{sj} = \operatorname{Re}\alpha_{sj}$ and, moreover, we set $c_{sj} = \operatorname{Im}\alpha_s\bar{\alpha}_k$, $s \neq j \neq k$. We note that $c_{sj} = -c_{kj}$. For brevity, we use the notation $c_{13} = \alpha$, $c_{21} = \beta$, $c_{12} = \gamma$.

From (1.11) for $a_{sj}^* = \operatorname{Re}\alpha_{sj}^*$ we obtain

$$a_{ss}^* = a_{ss}, \quad a_{sj}^* = a_{sj} + c_{sj}\delta^{-1}, \; s, j = 1, 2, 3, \quad s \neq j \tag{4.1}$$

To study system (1.11) in D^* we use Molchanov's theorem /7/ as it applies to system (1.8). For this, we write the model system corresponding to (1.8) in the variables ω_s^*:

$$\frac{1}{2}\omega_s^{\cdot *} = \omega_s^* \sum_{j=1}^{3} a_{sj}^*\omega_j^* \tag{4.2}$$

The system "far from resonance" is the following

$$\frac{1}{2}\omega_s^{\cdot *} = \omega_s^* \sum_{j=1}^{3} a_{sj}\omega_j^* \tag{4.3}$$

For (4.2) we introduce the matrix $A^* = (a_{sj}^*)$, the j-th principal minors A_j^* $(j = 1, 2, 3)$ and their determinants Δ^*, Δ_j^*. The notation A, A_j, Δ, Δ_j for system (4.3) has analogous meanings. By using (4.1) the determinants Δ^*, Δ_j^* can be written in the form

$$\begin{gathered}\Delta^* = \Delta + L\delta^{-1} + \Delta_0\delta^{-2} \\ \Delta_j^* = \Delta_j - (a_{sr}c_{rs} + a_{rs}c_{sr})\delta^{-1} - c_{sr}c_{rs}\delta^{-2}, \quad j \neq s \neq r \\ L = \sum_{s,\, j=1,\, s\neq j}^{3} c_{sj}A_{sj}, \; \Delta_0 = -\alpha\gamma\sum_{j=1}^{3} a_{j1} + \alpha\beta\sum_{j=1}^{3} a_{j2} - \beta\gamma\sum_{j=1}^{3} a_{j3}\end{gathered} \tag{4.4}$$

where A_{sj} is the minor of the element a_{sj} of the matrix A.

The following conditions separate out the fundamental case /2/, which we will indeed study:

$$a_{jj} \neq 0, \quad \alpha, \beta, \gamma \neq 0, \quad \Delta, \Delta_0, \Delta_j \neq 0, \quad j = 1, 2, 3 \; (\forall\mu \in D) \tag{4.5}$$

Making the conditions N and U specific. We consider system (1.7) for $\mu \in \Gamma_3$. Applying Theorem 3.1 from /2/, we ensure satisfaction of the condition N by the requirement that the following equalities be violated:

$$\operatorname{sign}\alpha = \operatorname{sign}\beta = \operatorname{sign}\gamma \tag{4.6}$$

We now present condition U. In conformity with the Molchanov's theorem, these conditions are reduced to ensuring that there are no neutral and unstable rays in the cone $K = \{\omega_j^* \geqslant 0\}$ in system (4.3). Conditions (4.5) ensure the absence of neutral rays. The absence of unstable rays will be ensured by seven groups of conditions B_j, C_j, E $(j = 1, 2, 3)$. Each group is a condition for no unstable rays in all the one-dimensional and two-dimensional faces of the cone K and within the cone K. Algebraically this means that the set of seven systems

$$a_{jj}q_j = 1, \quad j = 1, 2, 3 \tag{4.7j}$$
$$A_j q^{(j)} = l^{(j)}, \quad l^{(j)} = (1, 1), \quad q^{(j)} = (q_s, q_r), \quad s < r, \quad s, r \neq j \tag{4.8j}$$
$$Aq = l, \quad l = (1, 1, 1), \quad q = (q_1, q_2, q_3) \tag{4.9}$$

has no strictly positive solution. The asymptotic stability conditions (condition U) are thereby the following:

$$B_j\colon\ a_{jj} < 0, \quad j = 1, 2, 3$$
$$C_j\colon\ (\exists\ s \neq j)\ \ (\Delta_j^{(s)}\Delta_j < 0), \quad s, j = 1, 2, 3$$
$$E\colon\ (\exists\ s)\ (\Delta^{(s)}\Delta < 0), \quad s = 1, 2, 3$$

where $\Delta_j^{(s)}$, $\Delta^{(s)}$ are obtained from Δ_j, Δ by replacing the s-th column by $l^{(j)}$, l.

Asymptotic stability conditions for system (4.2) are formulated analogously. They are obtained by replacing all the determinants in conditions C_j and E by Δ^*, Δ_j^*, $\Delta^{*(s)}$, $\Delta_j^{*(s)}$. The conditions obtained for (4.2) will be denoted by B_j, C_j^*, E^* (conditions B_j do not change) We denote the corresponding systems of algebraic equations by (4.8_j^*), (4.9^*). The expressions for $\Delta_j^{*(s)}$, $\Delta^{*(s)}$ are the following

$$\Delta^{*(s)} = \Delta^{(s)} + L_s\delta^{-1} - 3c_{sr}c_{sj}\delta^{-2}, \quad \Delta_j^{*(s)} = \Delta_j^{(s)} - c_{sr}\delta^{-1} \tag{4.10}$$
$$L = c_{sr}(2a_{rj} - a_{sj} - a_{jj}) + c_{sj}(2a_{jr} - a_{sr} - a_{rr}), \quad s \neq j \neq r$$

Finding δ^*. We introduce the sets $H_0^\pm = \{\delta \gtrless 0 \mid \Delta^{*(s)}\Delta^* > 0,\ s = 1, 2, 3\}$, $H_q^\pm = \{\delta \gtrless 0 \mid \Delta_q^{*(s)}\Delta_q^* > 0, s \neq q\}$, $q = 1, 2, 3$ and their exact boundaries $m_i^\pm = \inf H_i^\pm$, $M_i^\pm = \sup H_i^\pm$, $i = 0, 1, 2, 3$.

For $\delta \in \bigcup\limits_{i=0}^{3} H_i^\pm$ the system is unstable. From conditions U it follows that all non-empty sets H_i^+ have upper bounds $(M_i^+ < +\infty)$, and H_i^- lower bounds $(m_i^- > -\infty)$. Everywhere below $s, q, j, r = 1, 2, 3$, $q \neq j \neq r$, $i = 0, 1, 2, 3$.

Assertion 1. If $H_0^+\,(H_0^-) \neq \varnothing$, then for any combination of signs α, β, γ different from (4.6) $m_0^+ > 0$ $(M_0^- < 0)$.

Indeed, let $\delta > 0$ and $H_0^+ \neq \varnothing$, to be specific. We see that $m_0^+ > 0$. Considering $\delta > 0$ to be sufficiently small, we obtain $\operatorname{sign} \Delta^{*(s)} = -\operatorname{sign} c_{sr}c_{sj}$ from (4.4). We see by direct substitution that when conditions (4.6) are violated there are a number of different signs among the $\Delta^{*(s)}$. Therefore, for sufficiently small δ the condition E^* is true. But then $\inf H_0^+ > 0$. The assertion is proved in the case when $\delta > 0$. The case when $\delta < 0$ is considered analogously.

The validity of the following assertion can be seen by direct substitution.

Assertion 2. Near resonance, for any combination of signs α, β, γ different from (4.6) and for any δ, a unique system (4.8_j^*) exists for which the condition C_j^* is violated.

Table 2 enables this system to be determined (enables the value of j to be found) from the known signs of the numbers $\alpha, \beta, \gamma, \delta$.

Table 2

№	$1^\pm$	$2^\pm$	$3^\pm$	$4^\pm$	$5^\pm$	$6^\pm$
α	+	−	−	+	+	−
β	+	−	+	−	−	+
γ	+	−	−	−	+	+
j $\delta > 0$	3	1	1	2	2	3
j $\delta < 0$	1	3	2	3	1	2

Here the superscipt (plus or minus) corresponds to the sign of δ.

Example. In system (1.1), $\alpha, \beta, \gamma, \delta > 0$ (1^+). Only system (4.8_3^*) has a positive solution. Condition C_3^* is violated.

To find δ^* we determine the number j from Table 2 for fixed $\alpha, \beta, \gamma, \delta$. We use the notation $I^\pm = \{\delta \gtrless 0\}$ and we introduce the sets $N_j^\pm = H_0^\pm \cup H_q^\pm \cup H_r^\pm$, $A^\pm = I^\pm \setminus (N_j^\pm \cup H_j^\pm)$.

The sets A^+, A^- are strictly separated from zero, as follows from Assertions 1 and 2, and it is clear by construction that $\delta^* = \inf A^+$ for $\delta > 0$ and $\delta^* = \sup A^-$ for $\delta < 0$.

It is seen from (4.4) and (4.10) that all the non-empty sets $H_i^\pm$ and $A^\pm$ are either intervals or a combination of a finite number of them. We make the value of δ^* specific (by confining ourselves to the case $\delta > 0$). If $M_j^+ < \min\{m_0^+, m_q^+, m_r^+\}$ or $M_j^+ \geqslant \max\{M_0^+, M_q^+, M_r^+\}$, then $\delta^* = M_j^+$. If both inequalities are violated and N_j^+ is an interval, then $\delta^* = \max\{M_i^+\}$. If N_j^+ is the union of several non-intersecting intervals, then several (not less than three) transitions hold of the type $H \to AY$, $AY \to H$ as $|\delta|$ grows. The last replacement of stability must have the form $H \to AY$, which is ensured by the conditions U.

Determination of the numbers $M_i^{\pm}, m_i^{\pm}$ reduces to solving linear and quadratic inequalities in δ, and it is not convenient to do this in general form.

As an illustration we present just the values of M_j^+ in the case 1^+. In this case $j=3$ and it is necessary to take account of (4.4) and (4.10) in the solution of the system of inequalities

$$\Delta_3^{-1}(\Delta_3^{(1)}\delta - \gamma)P^{-1}(\delta) \geqslant 0, \quad \Delta_3^{-1}(\Delta_3^{(2)}\delta - \beta)P^{-1}(\delta) \geqslant 0 \qquad (4.11)$$
$$(P(\delta) = \delta^2 - (a_{12}\beta + a_{21}\gamma)\Delta_3^{-1}\delta - \beta\gamma\Delta_3^{-1})$$

System (4.11) should be solved in six subcases corresponding to the conditions C_3 achieving the signs $\Delta_3, \Delta_3^{(1)}, \Delta_3^{(2)}$. The list of these cases and the corresponding values of M_3^+ are given in Table 3.

Table 3

№		1_1^+	1_2^+	1_3^+	1_4^+	1_5^+	1_6^+
Δ_3		$-$	$-$	$-$	$+$	$+$	$+$
$\Delta_3^{(1)}$		$+$	$-$	$+$	$+$	$-$	$-$
$\Delta_3^{(2)}$		$+$	$+$	$-$	$-$	$+$	$-$
M_3^+	$Q \leqslant 0$	$\delta_{2,3}$	δ_3	δ_2	$\delta_{1,2}$	$\delta_{1,3}$	δ_1
	$Q > 0$	$\delta_{1,3,3}$	$\delta_{1,2}$	$\delta_{1,2}$			

Here

$$\Delta_3 = a_{11}a_{22} - a_{12}a_{21}, \quad \Delta_3^{(1)} = a_{22} - a_{12}$$
$$\Delta_3^{(2)} = a_{11} - a_{21}, \quad \delta_1 = {}^1/_2\,(a_{12}\beta + a_{21}\gamma + \sqrt{Q})\,\Delta_3^{-1}$$
$$\delta_2 = \gamma(a_{22} - a_{12})^{-1}, \quad \delta_3 = \beta(a_{11} - a_{12})^{-1}, \quad \delta_{j,s} = \min(\delta_j, \delta_s)$$
$$Q = (a_{12}\beta + a_{21}\gamma)^2 + 4\beta\gamma\Delta_3$$

In cases $1_4^+ - 1_6^+$ always $Q > 0$.

5. Behaviour of the critical value of resonance detuning. "Danger" of resonance. The quantity δ^* can be considered as a quantitative characteristic of the danger of resonance. The greater the $|\delta^*|$, the stronger the influence of resonance on non-resonant systems, and the later the resonance instability is replaced by asymptotic stability. Safe resonance corresponds to cases when $\delta^* = 0$. It is seen from Sect.2 that one-frequency resonance is safe.

Let us write the resonance coefficients α_s in the form

$$\alpha_s = a_s + ib_s, \quad \sin\varphi_s = -a_s\,|\alpha_s|^{-1}, \quad \cos\varphi_s = b_s\,|\alpha_s|^{-1} \qquad (5.1)$$

Formulas (5.1) introduce the auxiliary angles φ_s, the resonance phases. These angles determine the difference $\varphi_s - \varphi_j$, the shift of the resonance phases, and are the structural parameters, by regulating which any value can be obtained for δ^* to increase or diminish the danger of resonance.

Two-frequency resonance. It is seen from Table 1 that the value of δ^* is directly proportional to the number $|A|$, which has the following form according to (5.1)

$$|A| = |\operatorname{Im}\alpha_1\bar{\alpha}_2| = |\alpha_1\alpha_2|\sin\Delta\varphi, \quad \Delta\varphi = \varphi_1 - \varphi_2 \qquad (5.2)$$

It is seen from (5.2) that as $\Delta\varphi \to 0, \pi$ we have $\delta^* \to 0$ and resonance becomes safe. The most dangerous resonance corresponds to the value $\Delta\varphi = \pm\pi/2$ for $A \gtrless 0$.

Three-frequency resonance. Introducing the resonance phases φ_s, we obtain

$$\alpha = |\alpha_1\alpha_2|\sin(\varphi_1 - \varphi_2), \quad \beta = |\alpha_2\alpha_3|\sin(\varphi_2 - \varphi_3), \quad \gamma = |\alpha_1\alpha_3|\sin(\varphi_1 - \varphi_3) \qquad (5.3)$$

We identify the angles φ_s with points of the unit circle. Condition N means /1/ that $\Delta\varphi_1\varphi_2\varphi_3$ is obtuse-angled or degenerate (at a point or chord not coincident with the diameter).

As φ_s changes so that $\Delta\varphi_1\varphi_2\varphi_3$ degenerates to a point of diameter, the resonance phase shifts tend to 0 or π, and then it is seen from (5.3) that $\alpha, \beta, \gamma \to 0$. By using (4.4) and (4.10) it can be seen that if $\alpha, \beta, \gamma \to 0$, then $\delta^* \to 0$. Consequently, the condition of degeneracy of $\Delta\varphi_1\varphi_2\varphi_3$ is the safety condition for resonance.

Remark. As is seen from Table 3, in certain cases $\delta^* \to 0$ if only one or two paraemters from the triplet α, β, γ tend to zero. In this case the safety condition for resonance can include the case of degeneration of a triangle into a chord and a right triangle.

The above discussion shows that in both cases considered $|\alpha_s|$ and the resonance phase shifts are structural parameters permitting the achievement that δ^* be a previously assigned number. That the resonance be safe can be achieved by selecting just the resonance phase shift.

6. Example. We consider the system of equation $(n = 2)$

$$z_s^{\cdot\cdot} + \nu_s^2(\mu) z_s = Z_s^{(2)}(\mu, z, z^{\cdot}) + Z_s^{(3)}(\mu, z, z^{\cdot}) + \ldots \tag{6.1}$$

Following /2/ we write the forms $Z_s^{(2)}, Z_s^{(3)}$ in the form

$$Z_s^{(2)} = \sum_{j,h=1}^{2} a_{jh}^{(s)} z_j z_h + b_{jh}^{(s)} z_j z_h^{\cdot} + c_{jh}^{(s)} z_j^{\cdot} z_h^{\cdot}, \quad Z_s^{(3)} =$$
$$\sum_{j,h,k=1}^{2} a_{jhk}^{(s)} z_j z_h z_k + b_{jhk}^{(s)} z_j z_h z_k^{\cdot} + c_{jhk}^{(s)} z_j z_h^{\cdot} z_k^{\cdot} + d_{jhk}^{(s)} z_j^{\cdot} z_h^{\cdot} z_k^{\cdot}$$

After substituting $x_s = z_s - i\nu_s^{-1} z_s^{\cdot}$ we obtain the following system in place in (6.1):

$$x_s^{\cdot} = i\nu_s(\mu) x_s + X_s^{(2)}(x, \bar{x}, \mu) + X_s^{(3)}(x, \bar{x}, \mu) + \ldots$$

The real parts of the coefficients of the forms $X_s^{(2)}\,(X_s^{(3)})$ are formed from the coefficients of the first group of terms: $z_j z_h^{\cdot}\ (z_j z_h z_k^{\cdot},\ z_j^{\cdot} z_h^{\cdot} z_k^{\cdot})$ and the imaginary parts from the coefficients of the second group of terms: $z_j z_h,\ z_j^{\cdot} z_h^{\cdot}\ (z_j z_h z_k,\ z_j z_h^{\cdot} z_k^{\cdot})$.

Let there be resonance $\nu_1(0) + 2\nu_2(0)$ in system (6.1). To satisfy condition N it is necessary that there be terms of both groups to second order in both equations of (6.1). In particular, it is sufficient that $a_{22}^{(1)}, b_{22}^{(1)}, b_{21}^{(2)}, c_{12}^{(2)} \neq 0$. We present below the result of a calculation of the coefficients of a continuous normal form under the assumption that all the coefficients to second order, except those mentioned, are zero:

$$\alpha_1 = -\frac{1}{4\nu_1}(b_{22}^{(1)}\nu_2 + ia_{22}^{(1)}), \quad \alpha_2 = \frac{1}{4\nu_2}(-b_{21}^{(2)}\nu_1 + ic_{12}^{(2)}\nu_1\nu_2) \tag{6.2}$$
$$a_{11} = \frac{1}{8}(b_{111}^{(1)} + 3d_{111}^{(1)}\nu_1^2), \quad a_{12} = \frac{1}{4}(-b_{221}^{(1)} + d_{122}^{(1)}\nu_2^2) +$$
$$\frac{m}{8\nu_2(\nu_1 - \nu_2)}, \quad a_{21} = \frac{1}{4}(b_{112}^{(2)} + d_{112}^{(2)}\nu_1^2)$$
$$a_{22} = \frac{1}{8}(b_{222}^{(2)} + 3d_{222}^{(2)}\nu_2^2) + \frac{m}{16\nu_2(\nu_1 - 2\nu_2)}, \quad m = a_{22}^{(1)}b_{21}^{(2)} - \nu_2^2 b_{22}^{(1)}c_{12}^{(2)}$$

It is seen from the expression for a_{sj} that conditions α or β can be satisfied for system (6.1). For instance, if

$$b_{111}^{(1)}, b_{222}^{(2)}, b_{112}^{(2)} < 0, \quad d_{111}^{(1)} = d_{112}^{(2)} = d_{222}^{(2)} = 0$$

then for all values $\mu \in D$ the condition α is satisfied (for $\nu_1 > 0, \nu_2 < 0$).

Under the conditions mentioned, we will have for $A\delta < 0$ (the first line of Table 1):

$$|\delta^*| = |Aa_{22}^{-1}|, \quad A = \nu_1(a_{22}^{(1)}b_{21}^{(2)} + b_{22}^{(1)}c_{12}^{(2)}\nu_2)$$

By using (6.2) we obtain for the resonance phase shift

$$\Delta\varphi = \arcsin(b_{22}^{(1)}c_{12}^{(2)}\nu_2^2 + a_{22}^{(1)}b_{21}^{(2)})\,|\alpha_1\alpha_2|^{-1}$$

For $A \gtrless 0$ the equality $\Delta\varphi = \pm\pi/2$ yields the conditions for which resonance is most dangerous.

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